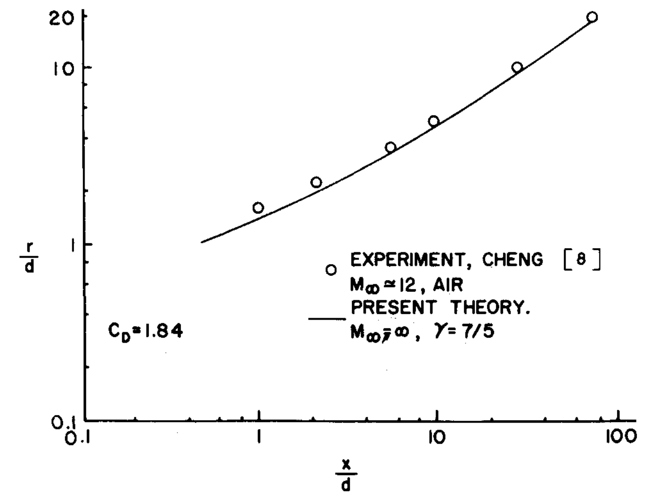

Fig. 1 Shock shape for a flat plate with a flat nose.

junction and set $V_{s_0} = V_b$. In order for C_p now to be finite at $x = 0$, we set $K_2 = 0$, and this also gives a finite slope to the shock at $x = 0$.

The integral in Eq. (19) now can be easily evaluated when $K_2 = 0$, and we obtain for axisymmetric bodies ($j = 1$)

$$x = \frac{\pi r_b^2}{K_1(2\varepsilon)^{1/2}}\left[z(2\varepsilon z^2+1)^{1/2}-(2\varepsilon+1)^{1/2}-\ln\frac{(2\varepsilon z^2)^{1/2}+(2\varepsilon z^2+1)^{1/2}}{(2\varepsilon)^{1/2}+(2\varepsilon+1)^{1/2}}\right] \quad (19)$$

where $z \equiv r_s/r_b$. For planar bodies ($j = 0$) we obtain

$$x = \frac{4r_b^{3/2}}{3K_1\varepsilon^{3/2}}\left[(\varepsilon z-1)(2\varepsilon z+1)^{1/2}-(\varepsilon-1)(2\varepsilon+1)^{1/2}\right] \quad (20)$$

For large z, these results take the same form as given by blast-wave theory. Near the nose, however, the finite thickness of the body is taken into account.

A comparison of Eq. (20) with the flat-plate data of Cheng et al.[8] is shown in Fig. 1, with $d = 2r_b$. The over-all agreement between experiment and theory is very good. The worst agreement is near the nose as might be expected. Both the data and theory show significant deviation from the straight-line behavior characteristic of first-order blast-wave results on log-log plots. A number of other comparisons are discussed in Ref. 1.

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Stability of a Spinning Satellite with Flexible Antennas

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Nomenclature

A, B, C	= principal moments of inertia of the rigid body about the x, y, z body axes, respectively
a_{ij}	= coefficients of quadratic terms in total kinetic energy expression
E	= Young's modulus
h	= distance from clamped end of beam to 0
I_x, I_y, I	= area moments of inertia of the elastic beam
l	= length of beam
M_t	= total system mass
0	= mass center of the complete system at dynamic equilibrium
p_k	= generalized momenta
q_k	= generalized coordinates
R_A	= ratio of mass moment of inertia of the pair of antennas about the x axis to A
T	= kinetic energy
T_R	= rotational kinetic energy
u_x, u_y	= displacement of a beam element relative to the rigid body
U	= dynamic potential
V	= potential energy of the system
v_c	= velocity of the system's mass center
x, y, z	= body axes coordinates
x_1, y_1, z_1	= body axes coordinates at dynamic equilibrium state
ψ, ϕ, θ	= Euler angles
ρ	= mass per unit length of elastic beam
π_1	= $\rho l/M_t$
$\pi_2{}^2$	= $EI/(\rho l^4)$
π_3	= $\rho l^3/A$
π_4	= h/l
π_5	= $\rho h^3/A$
$\bar{\pi}_3$	= $\rho l^3/B$
$\bar{\pi}_5$	= $\rho h^3/B$

Introduction

THE dynamic stability of spinning satellite systems with elastic components is an important space dynamics problem which has been receiving much attention in the recent literature. Because of the inherent analytical difficulties involved in studying such hybrid dynamical systems, many approximate techniques have evolved in the literature. The dynamic stability of flexible satellite systems has been studied by a wide variety of methods for several different configurations, with Liapunov's direct method being one of the more widely used techniques.

Recent papers by Meirovitch and Calico[1] and Barbera and Likens[2] present interesting discussions and comparisons between the various analytical methods available in the literature, which would make a further historical development redundant at this point. Of particular interest here is the use of Liapunov's direct method applied to elastic spacecraft, a subject which has been discussed by many researchers too numerous to mention. However, papers by Pringle,[3] Meirovitch,[4–6] Barbera and Likens,[2] Hughes and Fung,[7] Brown and Schlack,[8] and Kulla[9] are considered representative of these studies.

Received October 26, 1973; revision received July 1, 1974.

Index categories: Spacecraft Attitude Dynamics and Control; Structural Dynamic Analysis.

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When modeling an elastic element as an elastic continuum, a common procedure is to describe the shape of the elastic continuum by using a complete set of modal functions satisfying the boundary conditions of the problem. However, most investigators have found it necessary to truncate their assumed modal series before introducing it into the analysis for determining stability criteria. The difficulty of assessing the series truncation error using this procedure has led some investigators to become critical of modal analysis techniques when employing Liapunov's direct method.

Recently Brown and Schlack[8] presented a method for overcoming this difficulty by developing a technique for deriving closed-form solutions for the stability criteria in terms of an infinite series. In applying this method, the shape of the elastic continuum is expressed as an infinite series of modal terms, preferably satisfying both the natural and geometric boundary conditions, which are then carried through the entire Liapunov stability analysis. Using mathematical induction for evaluating the associated Hessian matrix, general stability criteria are subsequently obtained in the form of an infinite series of algebraic terms. Thus, the effect of series truncation can be easily studied by determining the stability boundaries for any number of terms n without resolving the entire problem for each value of n to be considered. In this way, very accurate solutions for the sufficient stability conditions based on the Hamiltonian function as a Liapunov testing function can be generated, yielding an excellent base for determining the accuracy of various approximate methods.

Physical Model and Modal Functions

The physical model of the system under consideration is shown in Fig. 1, with the angular rotations considered in the order ψ, ϕ, θ. Its mass center is assumed to orbit in a circular path and the system is considered to be free from external torque. In the undeformed state, the antennas lie along the z direction, which is the spin axis, and their elastic vibrations are assumed to take place in the transverse directions for which the deflections can be written as $\mathbf{u} = u_x \mathbf{i} + u_y \mathbf{j}$.

Modal fu· ctions describing the vibration of elastic antennas for a spinning system can be obtained[10] by solving the set of homogeneous Lagrange's equations given by

$$EI \frac{\partial^2 u_x}{\partial \bar{z}^2} + \rho \frac{\partial^2 u_x}{\partial t^2} - \rho \Omega^2 u_x - 2\Omega\rho \frac{\partial u_y}{\partial t} = 0 \tag{1}$$

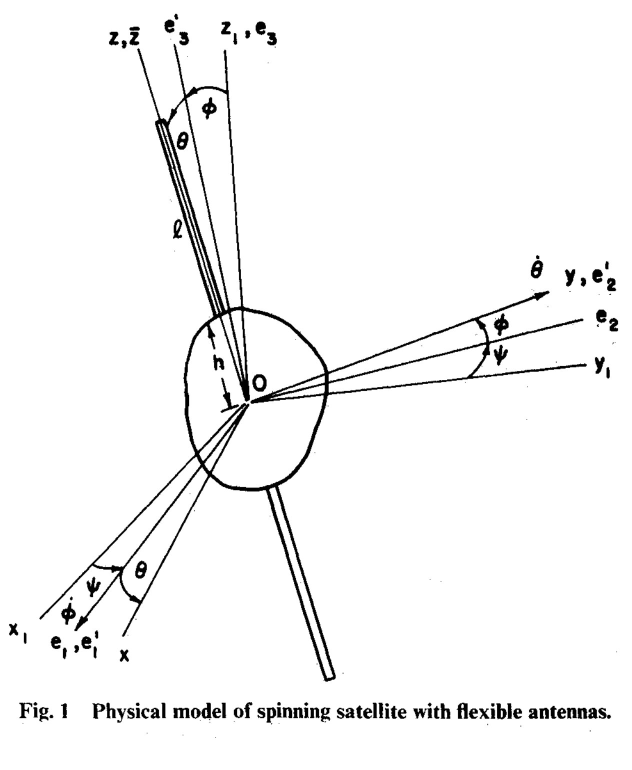

Fig. 1 Physical model of spinning satellite with flexible antennas.

$$EI \frac{\partial^2 u_y}{\partial \bar{z}^2} + \rho \frac{\partial^2 u_y}{\partial t^2} - \rho \Omega^2 u_y + 2\Omega\rho \frac{\partial u_x}{\partial t} = 0 \tag{2}$$

where the coordinate $\bar{z}$ has the same direction as the z axis, but has its origin at $z = h$. The boundary conditions are given by

$$u_x = 0, \ \frac{\partial u_x}{\partial \bar{z}} = 0, \ u_y = 0, \ \frac{\partial u_y}{\partial \bar{z}} = 0 \quad \text{at} \quad \bar{z} = 0 \tag{3}$$

$$\frac{\partial^2 u_x}{\partial \bar{z}^2} = 0, \ \frac{\partial^3 u_x}{\partial \bar{z}^3} = 0, \ \frac{\partial^2 u_y}{\partial \bar{z}^2} = 0, \ \frac{\partial^3 u_y}{\partial \bar{z}^3} = 0 \quad \text{at} \quad \bar{z} = l \tag{4}$$

The solution of Eqs. (1) and (2) consistent with boundary conditions (3) and (4) yields the following set of orthogonal modal functions, which incidentally are of the same form for either spinning or nonspinning classical beams:

$$u_x = \sum_{i=1}^{\infty} C_i f_i\left(\frac{\beta_i}{l} \bar{z}\right) \tag{5}$$

$$u_y = \sum_{i=1}^{\infty} E_i f_i\left(\frac{\beta_i}{l} \bar{z}\right) \tag{6}$$

where β_i are eigenvalues of the equation $1 + \cos\beta \cosh\beta = 0$ and

$$f_i\left(\frac{\beta_i}{l} \bar{z}\right) = (\sin\beta_i - \sinh\beta_i)\left(\sin\beta_i \frac{\bar{z}}{l} - \sinh\beta_i \frac{\bar{z}}{l}\right) + (\cos\beta_i + \cosh\beta_i)\left(\cos\beta_i \frac{\bar{z}}{l} - \cosh\beta_i \frac{\bar{z}}{l}\right) \tag{7}$$

Analysis

Following the analytical method developed by Brown and Schlack,[8] sufficient stability criteria can be established for the system by determining the conditions under which the dynamic potential U is positive definite, where

$$U = {p_1}^2/2a_{11} + V \tag{8}$$

In Eq. (8), p_1 is the generalized momentum corresponding to the cyclic coordinate q_1 and a_{11} is the coefficient in the rotational kinetic energy expression associated with the term $\dot{q}_1^{\,2} = \dot{\psi}^2$ where

$$T_R = T - \frac{1}{2} M_t v_c^{\,2} = \frac{1}{2} \sum_i \sum_j a_{ij} \dot{q}_i \dot{q}_j \tag{9}$$

The conditions under which the dynamic potential U is a positive definite function in the neighborhood of the dynamic equilibrium position E may be established by requiring that its Hessian matrix evaluated at E is positive definite, written mathematically as

$$[H]_E = \left[\frac{\partial^2 U}{\partial q_i \, \partial q_j}\right]_E = [b_{q_i,q_j}]_E > 0 \tag{10}$$

Application of Sylvester's criterion to the determinant of the Hessian matrix, along with the use of mathematical induction as demonstrated in Ref. 8, yields the following general, closed-form series solutions for the stability conditions in terms of dimensionless parameters, valid for any number of assumed modal terms n:

$$(C/A) - 1 - 2[\pi_5/\pi_4 + \pi_3\pi_4 + \tfrac{1}{3}\pi_3] > 0 \tag{11}$$

$$(C/B) - 1 - 2[\bar{\pi}_5/\pi_4 + \bar{\pi}_3\pi_4 + \tfrac{1}{3}\bar{\pi}_3] > 0 \tag{12}$$

$$(C/A) - 1 - 2[\pi_5/\pi_4 + \pi_3\pi_4 + \tfrac{1}{3}\pi_3] - \frac{\sum_{i=1}^{n} b_{\theta,i} |A_{\theta,i}| - \sum_{i''=1} b_{\theta,i''} |A_{\theta,i''}|}{A\Omega^2 |A_{\theta,\theta}|} > 0 \tag{13}$$

$$(C/B) - 1 - 2[\bar{\pi}_5/\pi_4 + \bar{\pi}_3\pi_4 + \tfrac{1}{3}\bar{\pi}_3] - \frac{\sum_{i'=1}^{n} b_{\phi,i'} |A_{\phi,i'}| - \sum_{\bar{\imath}=1}^{n} b_{\phi,\bar{\imath}} |A_{\phi,\bar{\imath}}|}{B\Omega^2 |A_{\phi,\phi}|} > 0 \tag{14}$$

The terms $b_{\theta,i}$, $b_{\phi,i}$, etc., in Eqs. (13) and (14) are the elements of the Hessian matrix written as Eq. (10) and the terms $A_{\phi,\phi}$,

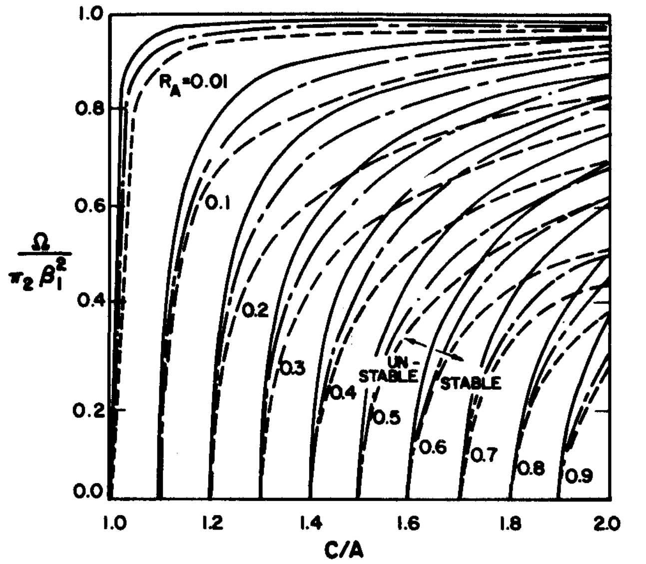

Fig. 2 Comparative study of stability regions, where ——— denotes results of present study, - - - - Meirovitch's results,[5] and ——·—— Meirovitch's results.[6]

$A_{\phi,i}$, etc., are the cofactors of the determinant of the Hessian matrix corresponding to the elements in the first row. The value of each determinant in Eqs. (13) and (14) can easily be calculated with the aid of a computer for any number of modal terms and the sums can be shown to be convergent. Note that as $n \to \infty$ the stability criteria given by Eqs. (11–14) provide an excellent means for studying the effects of modal series truncation on the accuracy of the stability region boundaries.

Discussion of Results and Conclusions

Computer results based on the stability conditions given by Eqs. (11–14) are presented in Fig. 2, where R_A is the ratio of the moment of inertia of the pair of antennas about the x axis to the moment of inertia of the rigid body about the same axis and $\Omega/\pi_2\beta_1{}^2$ is a dimensionless spin parameter. For R_A varying from 0.01 to 0.9 and $\Omega/\pi_2\beta_1{}^2$ varying from zero to 0.9, a single term in the modal series expansion yields results within 0.1% of those obtained with two or more modal terms. Thus, this analysis provides important evidence that the use of a single modal term satisfying both the geometric and natural boundary conditions of the problem yields surprisingly accurate stability region boundaries based on Liapunov's direct method even for rather flexible high-spin systems.

An identical mathematical model has been studied by Meirovitch[5] and the results are redrawn in Fig. 2 for purposes of comparison. Later, Meirovitch[6] developed a new and improved set of results for the same model which are also included in Fig. 2. It is easy to see that the present results contain larger stable regions than those of Refs. 5 and 6, with these differences magnifying as the spin rate and flexibility increase. These conclusions follow from the fact that Meirovitch's results are not developed directly from the Hamiltonian function, but are derived from a neighboring Liapunov testing function formed by adroitly employing the bounding properties of the Rayleigh quotient concept, thereby substantially reducing the algebraic complexities of the problem. The results presented herein serve as an important basis of comparison for assessing the accuracy of such approximate techniques, showing for this case that Meirovitch's approximation to the Hamiltonian gives conservative stability bounds. For a more detailed development of this research, see Ref. 11.

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Comment on the Equation for Aeroelastic Divergence of Unguided Launch Vehicles

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A MATRIX formulation of the problem of aeroelastic divergence of unguided launch vehicles using a discrete mass representation is presented in Ref. 1. The purpose of the present Note is to indicate an interesting feature of the equation of aeroelastic divergence. The final equation of aeroelastic divergence[1] is

$$\{F_n\} = q\overline{C_{N_\alpha}S}\left[\lceil 1 \rfloor + \left\{\frac{m_r}{M}\right\}\left\lfloor \frac{X_{cp}-X_r}{X_{cg}-X_{cp}} \right\rfloor + \left\lceil \frac{C_{N_\alpha}S_r}{\overline{C_{N_\alpha}S}} \right\rfloor \{1\} \left\lfloor \frac{X_r - X_{cg}}{X_{cg}-X_{cp}} \right\rfloor\right] \left\lceil \frac{C_{N_\alpha}S_r}{\overline{C_{N_\alpha}S}} \right\rfloor [\rho_{r,n}]\{F_n\} \quad \text{(1a)}$$

$$= q\overline{C_{N_\alpha}S}[A]\left\lceil \frac{C_{N_\alpha}S_r}{\overline{C_{N_\alpha}S}} \right\rfloor [\rho_{r,n}]\{F_n\} \quad \text{(1b)}$$

where

$$[A] = \left[\lceil 1 \rfloor + \left\{\frac{m_r}{M}\right\}\left\lfloor \frac{X_{cp}-X_r}{X_{cg}-X_{cp}} \right\rfloor + \left\lceil \frac{C_{N_\alpha}S_r}{\overline{C_{N_\alpha}S}} \right\rfloor \{1\} \left\lfloor \frac{X_r - X_{cg}}{X_{cg}-X_{cp}} \right\rfloor\right]$$

is transformation matrix (2)

[ ], ⌈ ⌋, { }, ⌊ ⌋ square, diagonal, column, and row matrices, respectively. $C_{N_\alpha}S_r$, F_r, m_r, and X_r are the product of normal-force-coefficient slope and panel area, total transverse force, mass, and distance from origin or reference station 0 of rth station, respectively.

Received January 15, 1974; revision received July 12, 1974.
Index categories: LV/M Structural Design (Including Loads); Aeroelasticity and Hydroelasticity.

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